# ON THE STABILITY OF LAMINAR BOUNDARY LAYER FLOW

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Please be informed that there are quite a number of printing errors in the Princeton Aero. Engr. Lab. Report No. 211 issued under contract N6 onr-27006, Project NR. 061-049

You will find enclosed an errorta sheet. Thank you.

Sin-I Cheng

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Aero. Engr. Dept.

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# ERRORTA SHEET

| PAGE NO. | EQ'N. NO. | LINE | ERROR |
|---|---|---|---|
| 3 | | 16 | |
| 5 | (7) | 23 | $\phi'_x + i\alpha\varphi$ |
| 8 | (15) | 17 | $\frac{\partial}{\partial x}(wf)$ |
| | | 18 | $i\sigma + i\sigma'_x$ |
| | | 18 | $\frac{M^2}{T} Z_4$ |
| | | 19 | $Z_{1xx} - 2i\alpha Z_{1x} - \alpha$ |
| | | 19 | $(Z_{4x} + i\alpha Z_4)$ |
| 9 | (16) | 3 | $i\sigma\, \sigma_x - \sigma\sigma'$ |
| | (17) | 14 | $\frac{M^2}{T} Z_7 - \frac{1}{T} Z_3$ |
| 15 | (33) | 6 | |
| | (36) | 19 | $+\left(\frac{1}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\chi_2^{(0)}$ |
| 16 | (38) | 10 | $+\frac{\sigma}{\gamma\nu_{1c}} \nu_0 \chi_6^{(0)}$ |
| 17 | | 19 | when $\chi_{5j}$ |
| | (43) | 21 | $\left(\frac{i\sigma'_c}{\nu_{1c}}\right)^{-2/3}$ |
| 18 | (44) | 10 | $\left(\frac{d}{dT}(\nu_1\mu)\right)_c$ |
| | | (11) | $-\frac{i\sigma'_c}{\nu_{1c}}\sigma(\quad)$ |
| 19 | | 4 | ,The |
| | | 7 | entical |

# ON THE STABILITY OF LAMINAR BOUNDARY LAYER FLOW [1]

by

Sin-I Cheng

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## List of Symbols

| Physical Coordinate | | Dimensionless Quantities | Characteristic Measure |
|---|---|---|---|
| **Positional Coordinates** | | | |
| 1. | $x^*$ | $x$ | $\delta^*$ |
| 2. | $y^*$ | $y$ | $\delta^*$ |
| **Time** | | | |
| 3. | $t^*$ | $t$ | $\delta^*/u_0^*$ |
| **Velocity components in the direction of x-axis and y-axis respectively** | | | |
| 4. | $u^*$ | $w(x,y)+f(x,y)e^z$ | $u_0^*$ |
| 5. | $v^*$ | $v(x,y)+\alpha\varphi(x,y)e^z$ | $u_0^*$ |
| **Density of gas** | | | |
| 6. | $\rho^*$ | $\rho(x,y)+r(x,y)e^z$ | $\bar{\rho}_0^*$ |
| **Temperature of gas** | | | |
| 7. | $T^*$ | $T(x,y)+\theta(x,y)e^z$ | $\bar{T}_0^*$ |
| **Pressure of gas** | | | |
| 8. | $p^*$ | $p(x,y)+\pi(x,y)e^z$ | $\bar{p}_0^*$ |
| **Coefficients of viscosity of gas** | | | |
| 9. | $\mu_1^*$ | $\mu_1(x,y)+m_1(x,y)e^z$ | $\bar{\mu}_{10}^*$ |
| 10. | $\mu_2^*$ | $\mu_2(x,y)+m_2(x,y)e^z$ | $\bar{\mu}_{10}^*$ |
| **Thermal conductivity of gas** | | | |
| 11. | $K^*$ | $\frac{1}{\sigma}\mu_1(x,y)+\frac{1}{\sigma}m_1(x,y)e^z$ | $c_p\bar{\mu}_{10}^*$ |
| **Wave number of the disturbance** | | | |
| 12. | $\alpha^*=\frac{2\pi}{\lambda^*}$ | $\alpha=\frac{2\pi}{\lambda}$ | $\delta^{*-1}$ |
| **Phase velocity of the disturbance** | | | |
| 13. | $c^*$ | $c=c_r+ic_i$ | $u_0^*$ |

Specific heat of gas at constant volume

14. $c_v^*$ $\quad$ 1 $\quad$ $c_v^*$

Specific heat of gas at constant pressure

15. $c_p^*$ $\quad$ $\gamma$ $\quad$ $c_v^*$

Reynolds Number of the flow

16. $$R_e = \bar{\rho}_0^* \bar{u}_0^* \delta^* / \mu_{10}^* = \bar{u}_0^* \delta^* / \nu_{10}^*$$

Mach Number of the flow

17. $$M = \bar{u}_0^* / \sqrt{\gamma R^* \bar{T}_0^*}$$

Prandtl Number of the flow

18. $$\sigma = c_p^* \bar{\mu}_1^* / K^*$$

19. $$\varepsilon = (\alpha R_e)^{-\frac{1}{3}}$$

20. $$\eta = \frac{y - y_c}{\varepsilon}$$

21. $$\frac{\partial}{\partial \xi} = R_e \frac{\partial}{\partial x}$$

## I. Introduction

The stability of two dimensional small disturbances in incompressible laminar boundary layer flow has been extensively studied by many authors on the assumption that boundary layer flows are essentially parallel flows, (references, 1, 2, 3, 4, and 5). Their results agree fairly well with the available experimental data, (reference 6). In references 7 and 8, the stability investigation is extended to the laminar boundary layer in a compressible fluid. The direct effect of the local pressure gradient on the calculation of the stability limits for incompressible boundary layer flow has been shown in reference 9 to be negligible under the approximation $\frac{1}{\alpha R_e} << 1$ , if the local velocity profile is used in the stability calculation. It has also been pointed out in reference 8 that the effect of the local pressure gradient in the compressible case can be expected to behave likewise. However, the assumption that the vertical velocity component in the boundary layer flow plays negligible role has not received careful attention.

It is questioned in reference 10 whether the boundary layer flow can be considered as a parallel flow and whether the gradients in the main stream direction have negligible effects in the investigation of the stability of the small disturbances in the laminar boundary layer. There is a qualitative argument that under the Prandtl boundary layer approximation, the variation of the mean flow properties in the x-direction within a few wave lengths of the disturbance is of the order of $\frac{1}{R_e}$ , which is negligibly small compared to unity. Therefore the contributions of such terms in the stability calculation can be neglected as higher order small quantities.

This kind of argument should be investigated more closely so far as the vertical velocity component is concerned even though the vertical velocity component is a small quantity of the order of $\frac{1}{R_e}$. The vertical velocity component produces a momentum transfer and an energy transfer across the boundary layer where both the disturbance quantities and the mean flow properties vary rapidly. The net effect of the transport processes may thus be much larger than the magnitude of the small agent that produces the transport processes. Although the vertical velocity component of the flow and the gradients of the flow properties in the x-direction are small quantities of the same order the net effect of the former in the stability calculation may be much more important than that of the latter. The present analysis verified this argument. It is shown that the vertical velocity component is the most critical factor that is neglected in previous analysis of the stability of the laminar boundary layer flow.

Because of the ingenious selection of the solutions of the disturbance amplitude functions made by the previous investigators, the vertical velocity component does not enter into the stability calculation in first approximation. In all these schemes the stability of the two dimensional small disturbances in laminar boundary layer flow is determined only by the local flow properties in first approximation. For higher approximations or for higher Mach numbers of the flow where the effect of vertical velocity is no longer small this statement is not correct.

## II. General Formulation

The system of differential equations for the disturbance functions is obtained from

(1) Equation of mass continuity

(2) Two equations of momentum, i.e., Navier Stokes equation in two dimensional form.

(3) Equation of energy balance

(4) Equation of state of an ideal gas

We shall take into account the variation of viscosity and thermal conductivity coefficients but neglect the variations of the specific heats with temperature in the boundary layer. Symbols are mostly adopted from reference 7 to facilitate comparison. Superscript (*) is used to indicate physical quantities and subscript (o) is used to denote free stream values of the quantities. Subscript (c) denotes that the quantity is evaluated at the critical layer where w = c.

Curvilinear coordinates are taken with x as the length along the wall and y normal to the wall, nondimensionalized by the thickness of the boundary layer. The dimensionless parameters are defined as

1. Reynolds Number $R_e = \bar{\rho}_o^* \bar{u}_o^* \delta^* / \mu_{1o}^*$

2. Mach Number $M = \bar{u}_o^* / \sqrt{\gamma R^* T_o^*}$

3. Prandtl Number $\sigma = c_p^* \mu_{1o}^* / K^*$

4. Froude Number is assumed to be infinitely large so that the effect of the acceleration due gravity is neglected.

The basic equations for the stability investigation in dimensionless form are:

continuity: $$\rho_t + (\rho u)_x + (\rho v)_y = 0 \tag{1}$$

1st momentum $$\rho u_t + \rho u u_x + \rho v u_y$$

$$= -\frac{1}{\gamma M^2} p_x + \frac{1}{Re}\left[2(\mu_1 u_x)_x + \frac{2}{3}\{(\mu_2-\mu_1)(u_x+v_y)\}_x + \{\mu_1(u_y+v_x)\}_y\right] \tag{2}$$

2nd momentum: $$\rho v_t + \rho u v_x + \rho v v_y$$

$$= -\frac{1}{\gamma M^2} p_y + \frac{1}{Re}\left[\{\mu_1(u_y+v_x)\}_x + 2(\mu_1 v_y)_y + \frac{2}{3}\{(\mu_2-\mu_1)(u_x+v_y)\}_y\right] \tag{3}$$

Energy

$$\rho\left[T_t + uT_x + vT_y\right]$$
$$= -(\gamma-1)(p u_x + p v_y)$$
$$+ \frac{\gamma(\gamma-1)M^2}{Re}\left[2\mu_1 u_x^2 + \frac{2}{3}(\mu_2-\mu_1)(u_x+v_y)u_x + 2\mu_1 v_y^2 + \frac{2}{3}(\mu_2-\mu_1)(u_x+v_y)v_y + 2\cdot\frac{1}{2}\mu_1(u_y+v_x)^2\right]$$
$$+ \frac{\gamma}{\sigma Re}\left[(\mu_1 T_y)_y + (\mu_1 T_x)_x\right]. \tag{4}$$

State $$p = \rho T \tag{5}$$

Here subscript x or y means partial differentiation with respect to x or y.

$\mu_1$ is the first viscosity coefficient and $\mu_2$ is the second or the bulk modulus of viscosity, and $\mu_2$ will be zero if the Stokes' relation is valid. With the assumption that $\mu^*$ and $K^*$ are functions of temperature only and that both $C_p^*$ and $\sigma$ are constant, we know that $\mu$ and $k$ in dimensionless form are identical. Thus in the energy equation $k$ is replaced by $\mu$ and both the variations of $k$ and $\mu$ are given by the variations of temperature multiplied by $\frac{d\mu_1}{dT}$.

In equations (1) to (5) replace each of the oscillating quantities by the sum of the time independent mean quantity and a small oscillation part of the quantity. For convenience in writing, the mean quantity is represented by the same letter. For example, we write for $\rho(x,y,t)$ the sum of $\rho(x,y) + r(x,y)\,e^{z}$ where $z = i\alpha(x-ct)$ with $\alpha$ = wave number of the oscillation = $\dfrac{2\pi}{\text{wave length}}$ and $c = c_r + c_i$ and both $c_r$ and $c_i$ are real. The amplitude functions $r(x,y)$ is time independent. $c_r$ stands for the velocity of the wave as observed by an observer on the wall, and $c_i$ is related to the amplification factor so that $c_i \gtrless 0$ determines whether the disturbance is amplified, neutral or damped.

As the amplitudes of the small perturbation quantities are assumed small, we shall neglect all the terms containing the product or square of these small perturbation amplitudes. Those terms containing only the mean flow quantities satisfy the system of equations (1) to (5) by themselves and therefore drop out of the disturbance equations. We then obtain the following linearized system of differential equations for the perturbation amplitude functions.

**Continuity:**

$$
\begin{aligned}
&(\varphi' + i f) + \frac{v}{\alpha}\frac{r'}{\rho} \\
&\quad = -i(w-c)\frac{r}{\rho} - \frac{\rho'}{\rho}\varphi - \frac{r}{\rho}\frac{v'}{\alpha} - \frac{1}{\alpha\rho}\frac{\partial}{\partial x}(\rho f + w r)
\end{aligned}
\tag{6}
$$

**First Momentum:**

$$
\begin{aligned}
&\alpha\rho\left[i(w-c)f + w'\varphi\right] + \rho\left[f w_x + w f_x + v f'\right] + \left[w w_x + v w'\right] r \\
&= -\frac{1}{\gamma M^2}(i\alpha\pi + \pi_x) \\
&\quad + \frac{1}{Re}\Big[\left(\tfrac{4}{3}\mu_1 + \tfrac{2}{3}\mu_2\right)\left(f_{xx} + 2i\alpha f_x - \alpha^2 f\right) + \left(\tfrac{\mu_1}{3} + \tfrac{2}{3}\mu_2\right)\alpha\left(\varphi_x' + i\alpha\varphi\right) + \mu_1 f''\Big] \\
&\quad + \frac{1}{Re}\frac{d\mu_1}{dT}\Big[\left(\tfrac{4}{3} + \tfrac{2}{3}\tau\right)T_x\left(f_x + i\alpha f\right) + \tfrac{2}{3}(\tau - 1)T_x\,\alpha\varphi' + \alpha T'\left(\varphi_x + i\alpha\varphi\right) + T' f'\Big] \\
&\quad + \frac{1}{Re}\frac{d\mu_1}{dT}\Big[\Big\{\left(\tfrac{4}{3} + \tfrac{2}{3}\tau\right)w_x + \tfrac{2}{3}(\tau - 1)v'\Big\}\left(\theta_x + i\alpha\theta\right) + \left(v_x + w'\right)\theta'\Big] \\
&\quad + \frac{1}{Re}\frac{d\mu_1}{dT}\Big[\left(\tfrac{4}{3} + \tfrac{2}{3}\tau\right)w_{xx} + w'' + \left(\tfrac{1}{3} + \tfrac{2}{3}\tau\right)v_x'\Big]\theta
\end{aligned}
\tag{7}
$$

**Second Momentum:**

$$\alpha^2\rho\left[i(w-c)\varphi\right] + \alpha\rho(\varphi v' + v\varphi') + \rho v_x f + \alpha\rho w \varphi_x + (w v_x + v v')r$$

$$= -\frac{\pi'}{\gamma M^2}$$

$$+ \frac{\mu_1}{Re}\Big\{\left[f_x' + i\alpha f' + \alpha\varphi_{xx} + 2i\alpha^2\varphi_x - \alpha^3\varphi\right]$$

$$+ \left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\alpha\varphi'' + \frac{2}{3}\left(\frac{\mu_2}{\mu_1} - 1\right)(f_x' + i\alpha f')\Big\}$$

$$+ \frac{1}{Re}\frac{d\mu_1}{dT}\Big\{T_x(f' + \alpha\varphi_x + i\alpha^2\varphi) + \left(\frac{4}{3} + \frac{2}{3}\tau\right)T'\alpha\varphi'$$

$$+ \frac{2}{3}(\tau - 1)T'(f_x + i\alpha f)\Big\}$$

$$+ \frac{1}{Re}\frac{d\mu_1}{dT}\Big\{(v_x + w')(\theta_x + i\alpha\theta) + \left(\frac{4}{3} + \frac{2}{3}\tau\right)v'\theta' + \frac{2}{3}(\tau - 1)w_x\theta'\Big\}$$

$$+ \frac{1}{Re}\frac{d\mu_1}{dT}\Big\{v_{xx} + \left(\frac{4}{3} + \frac{2}{3}\tau\right)v'' + \left(\frac{1}{3} + \frac{2}{3}\tau\right)w_x'\Big\}\theta$$

(8)

**Energy:**

$$\alpha\rho\left[i(w-c)\theta + T'\varphi\right] + \rho\left[w\theta_x + v\theta' + T_x f\right] + (w T_x + v T')r$$

$$= -(\gamma-1)\left[p(f_x + i\alpha f + \alpha\varphi') + (w_x + v')\pi\right]$$

$$+ \mu_1\frac{\gamma(\gamma-1)M^2}{Re}\Big\{\left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)2\left[w_x(f_x + i\alpha f) + \alpha v'\varphi'\right]$$

$$+ \frac{4}{3}\left(\frac{\mu_2}{\mu_1} - 1\right)\left[w_x\alpha\varphi' + v'(f_x + i\alpha f)\right]$$

$$+ 2(w' + v_x)(f' + \alpha\varphi_x + i\alpha^2\varphi)\Big\}$$

$$+ \frac{\gamma(\gamma-1)M^2}{Re}\frac{d\mu_1}{dT}\Big[\left(\frac{4}{3} + \frac{2}{3}\tau\right)(w_x^2 + v'^2) + \frac{4}{3}(\tau - 1)w_x v'$$

$$+ 2\cdot\frac{1}{2}(w' + v_x)^2\Big]\theta$$

$$+ \frac{\gamma\mu_1}{\sigma Re}\left[\theta'' + \theta_{xx} + 2i\alpha\theta_x - \alpha^2\theta\right]$$

$$+ \frac{\gamma}{\sigma Re}\frac{d\mu_1}{dT}\left[T''\theta + T_{xx}\theta + 2T_x'(\theta_x + i\alpha\theta) + 2T'\theta'\right]$$

(9)

**State:**

$$\frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T} \qquad (10)$$

In all these equations the prime is used to indicate differentiation with respect to y and subscript x indicates differentiation with respect to x. $\tau$ denotes the ratio of $\frac{d\mu_1}{d\tau} / \frac{d\mu_1}{d\tau}$ . Variations of the viscosity coefficients have been replaced by the corresponding variations in temperature.

These disturbance equations (6) to (9) are partial differential equations with two independent spatial variables x and y. The time coordinate has been separated by investigating the stability of the periodic solutions of the exponential type $\exp[i\alpha(x-ct)]$. The variation of the disturbance has been separated into two parts, a fast varying part depending on the frequency of the disturbance and a slowly varying part depending on the decay or the growth of the amplitude of the oscillation. It is the slowly varying part that enters equations (6) to (10). In view of the fact that the length in the x direction is very much larger than the corresponding length in the y direction for the same order of magnitude of the change of these amplitude functions, we may consider all those x gradients as independent of x in the first approximation. In other words we can consider equations (6) to (9) as a set of ordinary differential equations with y as the only independent variable. Thus we have four linear homogeneous ordinary differential equations and an algebraic equation of state for the five variables $f, \varphi, \pi, r$ & $\theta$. The analytic nature of these equations is almost the same as that of reference 7 where the vertical velocity component and the gradients in the x direction are neglected. The only equation that has been slightly modified is the continuity equation where $r'$ is brought into the equation by the vertical velocity component so that the continuity equation becomes a first order ordinary differential equation of both $\varphi$ and $r$ instead of $\varphi$ alone. The equation of state (10) can be used to eliminate r and

$$\frac{r'}{\rho} = \left(\frac{\pi'}{p} - \frac{\theta'}{T}\right) - \frac{\pi}{p}\frac{T'}{T} + \frac{\theta}{T}\left(\frac{T'}{T} - \frac{\rho'}{\rho}\right) \tag{11}$$

Define the following quantities:

$$\begin{array}{llll} Z_1 = f & Z_2 = f' & Z_3 = \varphi & Z_4 = \varphi' \\ Z_5 = \theta & Z_6 = \theta' & Z_7 = \dfrac{\pi}{M^2} & Z_8 = \dfrac{\pi'}{M^2} \end{array} \tag{12}$$

Then by definition we have

$$\left\{\begin{array}{l} \dfrac{dZ_1}{dy} = Z_2 \\ \dfrac{dZ_3}{dy} = Z_4 \end{array}\right. \qquad \left\{\begin{array}{l} \dfrac{dZ_5}{dy} = Z_6 \\ \dfrac{dZ_7}{dy} = Z_8 \end{array}\right. \tag{13}$$

With equations (11) and (12) equation of continuity (9) becomes:

$$\begin{aligned} & Z_4 + i Z_1 \\ & = -\left[i(w-c) + \frac{v'}{\alpha}\right]\left(\frac{M^2}{p} Z_7 - \frac{1}{T} Z_5\right) \\ & \quad - \frac{\rho'}{\rho} Z_3 \\ & \quad - \frac{v}{\alpha}\left[\left(\frac{M^2}{p} Z_8 - \frac{1}{T} Z_6\right) - \frac{T'}{T}\frac{M^2}{p} Z_7 + \frac{1}{T}\left(\frac{T'}{T} - \frac{\rho'}{\rho}\right) Z_5\right] \\ & \quad - \frac{1}{\alpha\rho}\frac{\partial}{\partial x}\left[\rho Z_1 + \rho w\left(\frac{M^2}{p} Z_7 - \frac{1}{T} Z_5\right)\right] \end{aligned} \tag{14}$$

The first momentum equation becomes:

$$\begin{aligned} \frac{dZ_2}{dy} = & \frac{\alpha Re}{\mu_1}\left[\rho i(w-c) Z_1 + \rho w' Z_3 + i \frac{Z_7}{\gamma}\right] \\ & + \frac{\rho v Re}{\mu_1} Z_2 + \frac{\rho Re}{\mu_1}\frac{\partial}{\partial x}(wf) \\ & + \frac{Re}{\mu_1}(w w_x' + v w')\left(\frac{M^2}{T} Z_4 - \frac{\rho}{T} Z_5\right) + \frac{Re}{\gamma\mu_1} Z_{7x} \\ & - \left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\left(Z_{1xx} - 2i\alpha Z_{1x} - \alpha^2 Z_1\right) - \left(\frac{1}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\left(Z_{4x} + i\alpha Z_4\right) \\ & - \frac{d\ln\mu_1}{dT}\left[\left(\frac{4}{3} + \frac{2}{3}\tau\right) T_x \left(Z_{1x} + i\alpha Z_1\right) + \frac{2}{3}(\tau - 1) T_x \alpha Z_4 + \alpha T'\left(Z_{3x} + i\alpha Z_3\right) + T' Z_2\right] \\ & - \frac{d\ln\mu_1}{dT}\left[\left(\frac{4}{3} + \frac{2}{3}\tau\right) w_x + \frac{2}{3}(\tau - 1) v'\right]\left(Z_{5x} + i\alpha Z_5\right) \\ & - \frac{d\ln\mu_1}{dT}\left[(v_x + w') Z_6 + \left\{\left(\frac{4}{3} + \frac{2}{3}\tau\right) w_{xx} + w'' + \left(\frac{1}{3} + \frac{2}{3}\tau\right) v_x'\right\} Z_5\right] \end{aligned} \tag{15}$$

The second momentum equation becomes

$$\frac{1}{\gamma} Z_8 - \frac{\alpha\mu_1}{Re}\left[\left(\frac{4}{3}+\frac{2}{3}\frac{\mu_2}{\mu_1}\right)\frac{dZ_4}{dy} + \left(\frac{1}{3}+\frac{2}{3}\frac{\mu_2}{\mu_1}\right) i Z_2 - \alpha^2 Z_3\right] + \alpha^2\rho\left[i(\omega - c) Z_3\right]$$

$$= -\alpha\rho(v' Z_3 + v Z_4) - \rho v_x Z_1 - \alpha\rho\omega Z_{3x} - (\omega v_x - v v')\left[\frac{M^2}{T} Z_7 - \frac{\rho}{T} Z_5\right]$$

$$+ \frac{\mu_1}{Re}\left[\left(\frac{1}{3}+\frac{2}{3}\frac{\mu_2}{\mu_1}\right) Z_{2x} + 2 i\alpha^2 Z_{3x} + \alpha Z_{3xx}\right]$$

$$+ \frac{1}{Re}\frac{d\mu_1}{dT}\left[(v_x + \omega') i\alpha Z_5 + \frac{2}{3}(\tau - 1) T' i\alpha Z_1 + \left(\frac{4}{3}+\frac{2}{3}\tau\right) T'\alpha Z_4\right]$$

$$+ \frac{1}{Re}\frac{d\mu_1}{dT}\Big\{\left[T_x\left(Z_2 + \alpha Z_{3x} + i\alpha^2 Z_3\right) + \frac{2}{3}(\tau - 1) T' Z_{1x} + (v_x + \omega') Z_{5x}\right]$$

$$+ \left[\left(\frac{4}{3}+\frac{2}{3}\tau\right) v' + \frac{2}{3}(\tau - 1)\omega_x\right] Z_6$$

$$+ \left[v_{xx} + \left(\frac{4}{3}+\frac{2}{3}\tau\right) v'' + \left(\frac{1}{3}+\frac{2}{3}\tau\right)\omega_x'\right] Z_5\Big\} \qquad (16)$$

Substituting the expression for $Z_4 + i Z_1$ from equation (14) into the energy equation (9) we obtain.

$$\frac{dZ_6}{dy} = \frac{\sigma}{\gamma\mu_1}\alpha Re\left[\gamma\rho\left\{i(\omega - c) Z_5 + T' Z_3\right\} - (\gamma - 1)\left\{p' Z_3 + i(\omega - c) M^2 Z_7\right\}\right]$$

$$- \frac{\gamma - 1}{\gamma}\frac{\sigma}{\mu_1} Re\left[v\left\{(M^2 Z_8 - \rho Z_6) - \frac{T'}{T} M^2 Z_7 + \left(\frac{T'}{T} - \frac{\rho'}{\rho}\right)\rho Z_5\right\}\right.$$

$$\left. + v'(M^2 Z_7 - \rho Z_5) + T\frac{\partial}{\partial x}\left(\rho Z_1 + \frac{\omega M^2}{T} Z_7 - \frac{\omega\rho}{T} Z_5\right)\right]$$

$$+ \frac{\sigma}{\gamma\mu_1} Re\left[(\omega T_x + v T')\left(\frac{M^2}{T} Z_7 - \frac{1}{T} Z_5\right) + \rho\omega Z_{5x} + \rho v Z_6 + \rho T_x Z_1\right]$$

$$+ \frac{\gamma - 1}{\gamma}\frac{\sigma}{\mu_1} Re\left[(\omega_x + v') M^2 Z_7 + p Z_{1x}\right]$$

$$+ \alpha^2 Z_5 - 2 i\alpha Z_{5x} - Z_{5xx}$$

$$- \frac{d\ln\mu_1}{dT}\left[T'' Z_5 + 2T' Z_6 + 2T_x'(Z_{5x} + i\alpha Z_5) + T_{xx} Z_5\right]$$

$$- (\gamma - 1)\sigma M^2\left[\left\{\left(\frac{4}{3}+\frac{2}{3}\frac{\mu_2}{\mu_1}\right) 2\omega_x + \frac{4}{3}\left(\frac{\mu_2}{\mu_1} - 1\right) v'\right\}(Z_{1x} + i\alpha Z_1) + \frac{4}{3}\left(\frac{\mu_2}{\mu_1} - 1\right)\omega_x\alpha Z_4\right.$$

$$\left. + \left(\frac{4}{3}+\frac{2}{3}\frac{\mu_2}{\mu_1}\right) 2\alpha v' Z_4 + 2(\omega' + v_x)(Z_2 + \alpha Z_{3x} + i\alpha^2 Z_3)\right]$$

$$- (\gamma - 1)\sigma M^2\frac{d\ln\mu_1}{dT}\left[\left(\frac{4}{3}+\frac{2}{3}\tau\right)(\omega_x^2 + v'^2) + \frac{4}{3}(\tau - 1)\omega_x v' + 2\cdot\frac{1}{2}(\omega' + v_x)^2\right] Z_5$$

(17)

Equations (13) and (17) are the eight equations for the determination of the eight unknown quantities $z_i$ with $i = 1, 2, \ldots 8$. The continuity equation (14) is algebraic and linear.

## IV. Approximate Solutions and the Boundary Value Problem.

It is almost impossible to find the exact solutions of the system of equations (13) to (17). Approximate solutions can however be obtained based upon the fact that $\frac{1}{R_e}$ is a very small quantity in the boundary layer flow. According to the Prandtl boundary layer approximations, whenever this approximation applies, the order of magnitude of the dimensionless quantities of mean flow properties are:

$$
\begin{gathered}
w = O(1) \qquad p = O(1) \quad , \qquad \rho = O(1) \\
T = O(1) \quad , \qquad \frac{\partial}{\partial y}\,(\text{velocity}) = O(1) \\
\frac{\partial p}{\partial y} = O\left(\frac{1}{Re}\right) \qquad\qquad \frac{\delta^*}{L} = O\left(\frac{1}{Re}\right) \\
\frac{\partial}{\partial x} = \frac{\delta^*}{L} \cdot \frac{\partial}{\partial (x/L)} = \frac{1}{Re}\,\frac{\partial}{\partial \xi} \\
v = \frac{1}{\rho}\int_0^y \frac{\partial}{\partial x}(\rho w)\,dy = \frac{1}{Re}\,\frac{1}{\rho}\,\frac{\partial}{\partial \xi}\int_0^y \rho w\,dy
\end{gathered}
\tag{18}
$$

where $\frac{\partial}{\partial \xi}$ is the differentiation operator along the x-direction, and does not change the order of magnitude of the quantity on which the operation is performed.

The methods of solving the disturbance equations for the compressible problem and those for the incompressible problem are essentially the same. These methods utilize different forms of the series expansion in terms of some convenient parameter related to $(\alpha R_e)^{-1}$.

1. The series expansion of the solutions $Z_i$ in terms of $(\alpha R_e)^{-1}$ is the most obvious one. In the first approximation, the system of disturbance equations (13) to (17) can be reduced to a second order ordinary differential equation with a singular point at $y = y_c$ where $w(y_c) = c$. This equation

$$\frac{d}{dy}\left\{\frac{(w-c)\varphi' - w'\varphi}{T - M^2(w-c)^2}\right\} - \frac{\alpha^2(w-c)}{T}\varphi = 0 \tag{19}$$

is identical with the inviscid equation as given in reference 7. Therefore two independent solutions can be obtained from this asymptotic expansion. The higher order approximations can be obtained by successive quadratures.

2. Transform the dependent variables $Z_i$ by

$$Z_i = f_i \exp\left[(\alpha R_e)^{-\frac{1}{2}} \int g\, dy\right] \tag{20}$$

where g is a function independent of $\alpha R_e$, and $f_i$ is expanded into power series of $(\alpha R_e)^{-\frac{1}{2}}$. Four independent asymptotic solutions, other than the two obtained from equation (19) are obtained. The initial approximations are identical with those given in reference 7.

3. Expand the variables $Z_i$ in terms of the powers of $(\alpha R_e)^{-\frac{1}{3}} = \varepsilon$ and transform the independent variable from y to $\eta = \frac{y - y_c}{\varepsilon}$ where $w(y_c) = c$. Six sets of independent solutions can be obtained. The initial approximations are identical with those given in reference 7. This series solution is considered as convergent and is recognized as being able to give any degree of accuracy if sufficient number of terms are taken in this $\varepsilon$ series.

As is well known, the proper selection of the approximate solutions to be used in the boundary value problem in the incompressible case has been a matter of considerable dispute. It is outlined in reference 4 that any of the following schemes can be used.

1. All sets of solutions taken from the convergent $\varepsilon$ series.
2. All sets of solutions taken from the asymptotic series.
3. Select the proper asymptotic series solutions to replace the corresponding solutions obtained from $\varepsilon$ series. For example, the two inviscid solutions obtained from equation (19) can be used to replace $X_{i3}$ and $X_{i4}$, that is, the third and the fourth sets of the $\varepsilon$ series solutions.

We shall investigate the effect of the vertical velocity component and the gradients in the x-direction on the solutions of the disturbance equations (13) to (17) using the $\varepsilon$ series.

Define the parameters $\varepsilon = (\alpha R_e)^{-1/3}$ and $\eta = \dfrac{y - y_c}{\varepsilon}$ (21)

where $\omega(y_c) = c$. Since c is in general complex, $y_c$ is also a complex quantity.

Expand all the mean flow quantities in Taylor series about the critical point $y_c$ thus:*

$$\omega - c = \omega_c'(\varepsilon\eta) + \frac{\omega_c''}{2!}(\varepsilon\eta)^2 + \frac{\omega_c'''}{3!}(\varepsilon\eta)^3 + \cdots \tag{22}$$

$$\rho = \rho_c + \rho_c'(\varepsilon\eta) + \frac{\rho_c''}{2!}(\varepsilon\eta)^2 + \cdots \tag{23}$$

$$T = T_c + T_c'(\varepsilon\eta) + \frac{T_c''}{2}(\varepsilon\eta)^2 + \cdots \tag{24}$$

and

$$v = \alpha\varepsilon^3 \frac{1}{\rho}\int_0^y \frac{\partial}{\partial\xi}(\rho\omega)\,dy$$

$$= \alpha\varepsilon^3\left[v_0 + v_1(\varepsilon\eta) + \frac{v_2}{2}(\varepsilon\eta)^2 + \cdots\right]$$

where

$$v_0 = \frac{1}{\rho_c}\left[\frac{\partial(\rho\omega)_c}{\partial\xi}y_c - \frac{\partial(\rho\omega)_c'}{\partial\xi}\frac{y_c^2}{2} + \cdots\right]$$

$$v_1 = \frac{1}{\rho_c}\left[\frac{\partial(\rho\omega)_c}{\partial\xi} - \frac{\rho_c'}{\rho_c}v_0\right] \tag{25}$$

$$v_2 = \cdots$$

*As pointed out recently by C. C. Lin (reference 12) Taylor series expansions for the mean flow quantities about the critical point are not very suitable for high Mach number flows, (M > 2, 3), or, one might add, for any flow across which there are large variations of vorticity or temperature.

Here $v_0$, $v_1$ etc. are all quantities of the order of unity. If the flow is incompressible, the fluid density is constant and

$$v_0 = \frac{\partial w_c}{\partial \xi} y_c - \frac{\partial w_c'}{\partial \xi} \frac{y_c^2}{2} + \cdots\cdots$$

which is still of the order of unity.

As explained in the previous section the variation of the magnitude of the disturbance has been separated into two parts; the fast varying part due to the wave propagation and the slowly varying part of the amplitude of oscillation. It is the amplitude function that enters the disturbance equation. Therefore we can write

$$\frac{\partial}{\partial x} = \frac{1}{Re}\frac{\partial}{\partial \xi} = \alpha \varepsilon^3 \frac{\partial}{\partial \xi} \tag{26}$$

when the operator $\frac{\partial}{\partial \xi}$ will not change the order of magnitude of the quantity on which the operator is applied for both the mean flow quantities and the disturbance amplitudes. With these facts in mind and also the relation $\frac{\partial}{\partial y} = \frac{1}{\varepsilon}\frac{\partial}{\partial \eta}$, we can determine the order of magnitude of each term in equations (13) to (17) if proper series expansions for $Z_i$ in terms of $\varepsilon$ are defined and if $\alpha$ is of the order of unity. The following forms of the expansions of $Z_i$ in terms of $\varepsilon$ are found to be self-consistent.

$$\begin{aligned}
Z_1 &= f = X_1^{(0)} + \varepsilon X_1^{(1)} + \varepsilon^2 X_1^{(2)} + \cdots\cdots \\
\varepsilon Z_2 &= \varepsilon f' = X_2^{(0)} + \varepsilon X_2^{(1)} + \varepsilon^2 X_2^{(2)} + \cdots\cdots \\
\varepsilon^{-1} Z_3 &= \varepsilon^{-1} \varphi = X_3^{(0)} + \varepsilon X_3^{(1)} + \varepsilon^2 X_3^{(2)} + \cdots\cdots \\
Z_4 &= \varphi' = X_4^{(0)} + \varepsilon X_4^{(1)} + \varepsilon^2 X_4^{(2)} + \cdots\cdots \\
Z_5 &= \theta = X_5^{(0)} + \varepsilon X_5^{(1)} + \varepsilon^2 X_5^{(2)} + \cdots\cdots \\
\varepsilon Z_6 &= \theta' = X_6^{(0)} + \varepsilon X_6^{(1)} + \varepsilon^2 X_6^{(2)} + \cdots\cdots \\
\varepsilon^{-1} Z_7 &= \varepsilon^{-1}\frac{\pi}{M^2} = X_7^{(0)} + \varepsilon X_7^{(1)} + \varepsilon^2 X_7^{(2)} + \cdots\cdots \\
Z_8 &= \frac{\pi'}{M^2} = X_8^{(0)} + \varepsilon X_8^{(1)} + \varepsilon^2 X_8^{(2)} + \cdots\cdots
\end{aligned} \tag{27}$$

By substituting these series (27) and expansions (22) to (25) into equations (13) to (17) and equating coefficients of different powers of $\varepsilon$ on both sides of each equation, we obtain a system of linear ordinary differential equations from which successive order approximations can be found by quadratures.

Equation (13) gives the following simple relations

$$\left\{\begin{aligned} \frac{dX_i^{(0)}}{d\eta} &= X_{i+1}^{(0)} \\ \frac{dX_i^{(k)}}{d\eta} &= X_{i+1}^{(k)} \qquad i = 1, 3, 5 \text{ and } 7 \end{aligned}\right. \tag{28}$$

for the first and the $K+1^{th}$ approximations respectively.

The continuity equation (14) gives

First approximation of the order of $\varepsilon^0$

$$X_4^{(0)} + i X_1^{(0)} = 0 \qquad \text{or} \qquad \frac{dX_3^{(0)}}{d\eta} + i X_1^{(0)} = 0 \tag{29}$$

Second approximation of the order of $\varepsilon$

$$X_4^{(1)} + i X_1^{(1)} = \frac{dX_3^{(1)}}{d\eta} + i X_1^{(1)} = \frac{i\omega_c'}{T_c}\eta X_5^{(0)} - \frac{\rho_c'}{\rho_c} X_3^{(0)} \tag{30}$$

Third approximation of the order of $\varepsilon^2$

$$\begin{aligned} X_4^{(2)} + i X_1^{(2)} &= \frac{dX_3^{(2)}}{d\eta} + i X_1^{(2)} \\ &= \frac{i\omega_c'}{T_c}\eta X_5^{(1)} - \frac{\rho_c'}{\rho_c} X_3^{(1)} \\ &+ i\left[\frac{\omega_c''}{T_c}\frac{\eta^2}{2} - \frac{\omega_c' T_c'}{T_c^2}\eta^2\right] X_5^{(0)} - \left(\frac{\rho_c''}{\rho_c} - \frac{\rho_c'^2}{\rho_c^2}\right) X_3^{(0)} - i\omega_c' \frac{M^2}{\rho_c}\eta X_7^{(0)} \\ &+ v_0 \frac{1}{T_c} X_6^{(0)} \end{aligned} \tag{31}$$

The vertical velocity component begins to appear in the continuity equation at the third approximation of the order of $\varepsilon^2$. The next higher approximation of the order of $\varepsilon^3$ will bring in terms involving the gradients in the x-direction.

The first momentum equation (15) gives

First approximation of the order of $\varepsilon^0$

$$\frac{dx_2^{(0)}}{d\eta} - \frac{\omega_c'}{\nu_{1c}}\left[i\eta x_1^{(0)} + x_3^{(0)}\right] - \frac{i}{\gamma\mu_{1c}} x_7^{(0)} = 0 \qquad (32)$$

Second approximation of the order of $\varepsilon$

$$\frac{dx_2^{(1)}}{d\eta} - \frac{\omega_c'}{\nu_{1c}}\left[i\eta x_1^{(1)} + x_3^{(1)}\right] - \frac{i}{\gamma\mu_{1c}} x_7^{(1)}$$
$$= \left\{\frac{\omega_c''}{\nu_{1c}} + \frac{\omega_c'}{\nu_{1c}}\left[\frac{\rho_c'}{\rho_c} - \left(\frac{d\ln\mu_1}{dT}\right)_c T_c'\right]\right\}\eta\,(i\eta x_1^{(0)} + x_3^{(0)})$$
$$- \frac{i}{\gamma\mu_{1c}}\left(\frac{d\ln\mu_1}{dT}\right)_c T_c'\eta\, x_7^{(0)} - \left(\frac{d\ln\mu_1}{dT}\right)_c (\omega_c' x_6^{(0)} + T_c' x_2^{(0)})$$
$$+ \frac{v_0}{\nu_{1c}} x_2^{(0)} \qquad (33)$$

Both the vertical velocity component $v$ and the temperature sensitivity of the viscosity coefficient enter into the first momentum equation at the second approximation of the order of $\varepsilon$. The next higher order approximation $\varepsilon^2$ will bring in terms involving the gradients in the x-direction.

The second momentum equation (16) gives

First approximation of the order of $\varepsilon^0$

$$x_8^{(0)} = \frac{dx_7^{(0)}}{d\eta} = 0 \qquad (34)$$

Second approximation of the order of $\varepsilon$

$$x_8^{(1)} = \frac{dx_7^{(1)}}{d\eta} = 0 \qquad (35)$$

Third approximation of the order of $\varepsilon^2$

$$x_8^{(2)} = \gamma\alpha^2\mu_1\left[\left(\frac{4}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right)\frac{dx_4^{(0)}}{d\eta} + \left(\frac{1}{3} + \frac{2}{3}\frac{\mu_2}{\mu_1}\right) x_2^{(0)}\right.$$
$$\left. - \frac{i\omega_c'}{\nu_{1c}}\eta\, x_3^{(0)}\right] \qquad (36)$$

It is only in the fourth approximation of the order of $\varepsilon^3$ that the terms involving the vertical velocity component will appear, and the gradients in the x-direction will come in at the next higher approximation.

The energy equation (17) gives

First approximation of the order of $\varepsilon^0$

$$\frac{dX_6^{(0)}}{d\eta} - \frac{\sigma}{\nu_{1c}}\left[i\omega_c'\eta X_5^{(0)} + \left(T_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{\rho_c}\right)X_3^{(0)}\right] = 0 \tag{37}$$

Second approximation of the order of $\varepsilon$

$$\begin{aligned}\frac{dX_6^{(1)}}{d\eta} - \frac{\sigma}{\nu_{1c}}\left[i\omega_c'\eta X_5^{(1)} + \left(T_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{\rho_c}\right)X_3^{(1)}\right] \\ = \frac{\sigma}{\nu_{1c}}\Big\{\Big[\frac{\rho_c'}{\rho_c} - \left(\frac{d\ln\mu_1}{dT}\right)_c T_c'\Big]\left(i\omega_c'\eta^2 X_5^{(0)} + T_c'\eta X_3^{(0)}\right) \\ - \frac{\gamma-1}{\gamma}\Big[\frac{p_c''}{\rho_c} - \left(\frac{d\ln\mu_1}{dT}\right)_c T_c'\frac{p_c'}{\rho_c}\Big]\eta X_3^{(0)} \\ + i\omega_c''\frac{\eta^2}{2}X_5^{(0)} + T_c''\eta X_3^{(0)}\Big\} \\ - \frac{\gamma-1}{\gamma}\frac{\sigma}{\mu_{1c}} i\omega_c'\eta M^2 X_7^{(0)} - 2\sigma(\gamma-1)M^2\omega_c' X_2^{(0)} - \left(\frac{d\ln\mu_1}{dT}\right)_c 2T_c' X_6^{(0)} \\ + \frac{\sigma}{\gamma\nu_{1c}}\cdot V_0 X_6^{(0)}\end{aligned} \tag{38}$$

The vertical velocity component enters the energy equation at the second approximation of the order of $\varepsilon$. The gradients in the x-direction will appear in the next higher order approximation.

For the first approximation of the order of $\varepsilon^0$ we have the following set of differential equations.

$$\left\{\begin{aligned} &\frac{dX_3^{(0)}}{d\eta} = -iX_1^{(0)} && (29)\\ &\frac{d^2X_1^{(0)}}{d\eta^2} = \frac{\omega_c'}{\nu_{1c}}\left(i\eta X_1^{(0)} + X_3^{(0)}\right) + \frac{i}{\gamma\mu_{1c}}X_7^{(0)} && (32)\\ &\frac{dX_7^{(0)}}{d\eta} = 0 && (34)\\ &\frac{d^2X_5^{(0)}}{d\eta^2} = \frac{\sigma}{\nu_{1c}}\left[i\omega_c'\eta X_5^{(0)} + \left(T_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{\rho_c}\right)X_3^{(0)}\right] && (37)\end{aligned}\right.$$

These equations are identical with those given in reference 7. By eliminating $X_3^{(0)}$ from equations (29) and (32) one obtains a differential equation for $\frac{dX_1^{(0)}}{d\eta}$ as follows:

$$\frac{d^3X_1^{(0)}}{d\eta^3} - \frac{i\omega_c'}{\nu_{1c}}\eta\frac{dX_1^{(0)}}{d\eta} = 0 \tag{39}$$

whose solutions are known in terms of the Hankel functions of the first and the second kind of the order 1/3, with argument $z = \frac{2}{3}(i\zeta)^{3/2}$, where $\zeta = \left(\frac{\omega_c'}{\nu_{1c}}\right)^{1/3}\eta$.

The six sets of independent solutions are obtained as:

$$\left\{\begin{aligned} X_{11}^{(0)} &= \int H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{1/2}\,d\zeta \\ X_{12}^{(0)} &= \int H_{1/3}^{(2)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{1/2}\,d\zeta \\ X_{13}^{(0)} &= 1 \\ X_{14}^{(0)} &= X_{15}^{(0)} = X_{16}^{(0)} = 0 \end{aligned}\right. \tag{40}$$

and equation (29) gives after integration,

$$\left\{\begin{aligned} X_{31}^{(0)} &= -i\left(\frac{\omega_c'}{\nu_{1c}}\right)^{-1/3}\left\{\zeta\int H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{1/2}d\zeta - \int H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{3/2}d\zeta\right\} \\ X_{32}^{(0)} &= -i\left(\frac{\omega_c'}{\nu_{1c}}\right)^{-1/3}\left\{\zeta\int H_{1/3}^{(2)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{\frac{1}{2}}d\zeta - \int H_{1/3}^{(2)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{3/2}d\zeta\right\}, \\ X_{33}^{(0)} &= -i\left(\frac{\omega_c'}{\nu_{1c}}\right)^{-1/3}\zeta \\ X_{34}^{(0)} &= 1 \\ X_{35}^{(0)} &= X_{36}^{(0)} = 0 \end{aligned}\right. \tag{41}$$

Equation (37) can be used to solve for $X_{5j}^{(0)}$. For $j = 5, 6$, $X_{35}^{(0)} = X_{36}^{(0)} = 0$ and equation (37) becomes homogeneous and

$$\left\{\begin{aligned} X_{55}^{(0)} &= \zeta^{1/2} H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\sigma^{\frac{1}{2}}\right] \\ X_{56}^{(0)} &= \zeta^{1/2} H_{1/3}^{(2)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\sigma^{\frac{1}{2}}\right] \end{aligned}\right. \tag{42}$$

When $X_{5j}$ with $j = 1, 2, 3$ and 4 can be obtained by quadrature. The formula for quadrature is

$$X_{5j}^{(0)} = \frac{\sigma}{\nu_{1c}}\left(\tau_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{p_c}\right)\frac{\pi}{6}\left[X_{56}^{(0)}\int X_{55}^{(0)}X_{3j}^{(0)}\,d\eta - X_{55}^{(0)}\int X_{56}^{(0)}X_{3j}^{(0)}\,d\eta\right] i\left(\frac{\omega_c'}{\nu_{1c}}\right)^{-\frac{2}{3}} \tag{43}$$

For the second approximation, equations (30), (33), (35) and (38) will be used. The homogeneous parts of each of these equations is the same as the corresponding equation for the first approximation. The inhomogeneous part are known functions involving the solutions of the previous approximations and the local velocity and the local temperature profiles and so forth. Therefore the second approximations can be obtained from the first approximations by quadrature with a formula analogous to equation(43). For example, if we evaluate $X_{14}^{(1)}$, we differentiate equation (33) and use equation (30) to eliminate $\frac{dX_{34}^{(1)}}{d\eta}$. We obtain the differential equation for $X_{14}^{(1)}$ whose homogeneous part is the same as equation (39):

$$\begin{aligned}
&\frac{d^3X_{14}^{(1)}}{d\eta^3} - \frac{i w_c'}{\nu_{1c}}\eta\frac{dX_{14}^{(1)}}{d\eta} \\
&= \frac{w_c''}{\nu_{1c}} + i\frac{w_c'^2}{\nu_{1c}}\eta\frac{X_{54}^{(0)}}{T_c} - i\sigma\frac{w_c'^2}{\nu_{1c}}\eta\left(\frac{d\ln\mu}{dT}\right)_c X_{54}^{(0)} \\
&- \left(\frac{d\ln\mu_1}{dT}\right)_c\left[\frac{w_c' T_c'}{\nu_{1c}} + \frac{iT_c'}{\gamma\mu_{1c}}X_{74}^{(0)} - \frac{w_c'}{\nu_{1c}}\sigma\left(T_c' - \frac{\gamma-1}{\gamma}\frac{p_c'}{\rho_c}\right)\right]
\end{aligned} \tag{44}$$

In obtaining equation (44) the following values of the disturbance functions have been introduced:

$$X_{14}^{(0)} = 0\ , \qquad X_{24}^{(0)} = 0\ , \qquad \text{and} \qquad X_{34}^{(0)} = 1$$

and $X_{54}^{(0)}$ is to be obtained from equation (43) with $j = 4$.

Denote the right hand side of equation (44) by $L_{14}^{(1)}(\eta)$ the following formula can be used to obtain $X_{14}^{(1)}$

$$X_{14}^{(1)} = \frac{i\pi}{6}\left(\frac{w_c'}{\nu_{1c}}\right)^{-1/3}\int\left\{\frac{dX_{12}^{(0)}}{d\eta}\int\frac{dX_{11}^{(0)}}{d\eta}L_{14}^{(1)}(\eta)\,d\eta - \frac{dX_{11}^{(0)}}{d\eta}\int\frac{dX_{12}^{(0)}}{d\eta}L_{14}^{(1)}(\eta)\,d\eta\right\}d\eta \tag{45}$$

The explicit form of $X_{14}^{(1)}$ must contain a logarithmic term when $\eta$ is large. Qualitatively, when $\eta$ is large, the term $\frac{d^3X_{14}^{(1)}}{d\eta^3}$ gives negligible contributions as compared to $\eta\frac{dX_{14}^{(1)}}{d\eta}$ and the solution of equation (44) will behave like the solution of

$$\eta\frac{dX_{14}^{(1)}}{d\eta} = i\frac{\nu_{1c}}{w_c'}L_{14}^{(1)} \tag{46}$$

Thus $\chi_{14}^{(1)}$ must contain this important term

$$i \frac{v_{1c}}{w_c'} h_{14}^{(1)} [\ln (y-y_c) - \ln \varepsilon] \tag{47}$$

The coefficient of this logarithmic term $i \frac{v_{1c}}{w_c'} h_{14}^{(1)}$ is a complicated known function of the local velocity profile, The local temperature profile and $\left(\frac{d \ln \mu_1}{dT}\right)_c$, but does not depend on the vertical velocity component. $\chi_{34}^{(1)}$ is obtained by integrating equation (45). The logarithmic term will appear in $\chi_{34}^{(1)}$ as $\eta \ln \eta$ and will vanish near the entical layer $y \to y_c$ or $\eta \to 0$. The inviscid solution $\phi_2$ which is used to replace this solution $\chi_{34}^{(0)} + \varepsilon \chi_{34}^{(1)}$ has been shown in reference 7 to approach zero as $(y-y_c) \ln (y-y_c)$ when y approaches $y_c$. The correspondence of these two solutions are thus made clear.

The same procedure can be used to find $\chi_{3j}^{(1)}$ with $j = 1, 2, 3, 5$ and 6. It is to be noticed that the vertical velocity component enters into these calculations only through the term $\frac{1}{v_{1c}} v_0 \cdot \chi_{2j}^{(0)}$. Since $\chi_{2j}^{(0)} = \frac{d \chi_{1j}^{(0)}}{d\eta} = 0$ for all values of j except j = 1, and 2, it follows that all these functions $\chi_{33}^{(1)}$, $\chi_{34}^{(1)}$ $\chi_{35}^{(1)}$ and $\chi_{36}^{(1)}$ are independent of the vertical velocity component. But $\chi_{31}^{(1)}$ and $\chi_{32}^{(1)}$ do depend on the vertical velocity component for both the compressible and the incompressible boundary layer flow.

The local gradients of pressure and temperature in the x-direction do not enter into the evaluation of these disturbance amplitude functions at the second approximation of the order of $\varepsilon$. The gradients in the x direction enter only when we go to the order of $\varepsilon^2$. Therefore the effect of the vertical velocity component is more critical than the effect of the gradients in the x-direction in the stability calculation. The effect of the vertical velocity component should be taken into account before the effect of the local pressure gradient in the main stream could be considered.

So far as the effect of the vertical velocity component on the determination of the stability boundary is concerned, the different selections of the disturbance functions to be used in the boundary value problem are of considerable importance. Suppose we take all $\chi_{3j}$ from the $\varepsilon$ series and consistently take all six solutions to the order of $\varepsilon$ i.e. $\chi_{3j}^{(0)} + \varepsilon \chi_{3j}^{(1)}$ in the boundary value problem, then one sees that the dependence of the solutions on the vertical velocity component through $\chi_{31}^{(1)}$ and $\chi_{32}^{(1)}$ will not be consistent with the simplification of assuming the boundary layer flow as a parallel flow.

Fortunately, all the previous investigators are satisfied with the first approximation of the two "viscous solutions" $\chi_{31}^{(0)}$ and $\chi_{32}^{(0)}$, while they used $\chi_{33}^{(0)} + \varepsilon \chi_{33}^{(1)}$ and $\chi_{34}^{(0)} + \varepsilon \chi_{34}^{(1)}$ <u>or</u> the two equivalent inviscid solutions, <u>or</u> the inviscid solutions corrected for viscosity in the boundary value problem. Thus the stability boundary as determined by any of these methods will be independent of the vertical velocity component and their results are consistent with the assumption that boundary layer flows are essentially parallel flows. But the accuracy of the quantitative determination of the stability boundary as carried out in references 8 and 14 can not be improved by taking more terms in the $\varepsilon$ series without including the effect of the vertical velocity component.

Unfortunately, in some practical cases, the parameter $\varepsilon$ is not a very small quantity near the minimum critical Reynolds number based on boundary layer thickness. The Reynolds number may be only 1000 while $\alpha$ is about unity. Thus $\varepsilon$ is only 1/10. In such cases it may be necessary for accurate quantitative determination of the stability limit and the amplification rates to go to the next approximation, in which case the effect of the vertical velocity component must be included.

In addition, at very high Mach numbers, the vertical velocity component in the boundary layer is of the order of $M^4/Re_{\delta^*}$ (reference 13) and may enter the stability problem of the laminar boundary layer even in first approximation. This question requires careful investigation before any statements can be made about the stability of the hypersonic laminar boundary layer.

## IV. Conclusions

1. The vertical velocity component is the most critical factor that is neglected in previous stability investigations of the laminar boundary layer flow. It is justifiable to consider the boundary layer flow as parallel flow and neglect $v$ only when the viscous solutions $\chi_{i1}$ and $\chi_{i2}$ are taken to be the first approximation of the order of $\varepsilon^0$ in the $\varepsilon$ series as is done by all the previous investigators.
2. The local pressure gradient and the local temperature gradient in the x-direction are less critical than the vertical velocity component in the determination of the stability boundary. These local gradients in the x-direction enter only to the order of $\varepsilon^2$ or $(\alpha R_e)^{-2/3}$.
3. The stability of the laminar boundary layer is determined only by the <u>local</u> flow properties for both the compressible and the incompressible flow within the order of approximation attempted by previous investigators. In other words the local pressure gradient and the local temperature gradient and the vertical velocity component of the boundary layer flow will not affect the calculation of the stability boundary in the first approximation, provided

that the local velocity profile and the local temperature profile are used in the stability calculation. These local velocity and local temperature profiles are of course intimately connected with the history of the upstream pressure gradient and the heat transfer conditions along the wall. As is pointed out in reference 9, this conclusion is of great practical importance for the determination of the beginning point of the instability of the laminar boundary layer flow over an airfoil, and also for the calculation of the rates of growth of the small disturbances downstream of the stability limit within the framework of the linearized small perturbation theory.

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